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THE WEAK STABILIZABILITY OF LINEAR SYSTEMS IN HILBERT SPACE

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ABSTRACT.

We will study the problem of stabilizing infinite dimensional linear control systems by output feedback. Describing such a system in the framework of Hilbert space and employing the semigroup representation of the system, we will show, under certain assumptions, a necessary and sufficient condition for the weak stabilizability.

I. Introduction.

The problem of stabilizing dynamical systems by feedback has a very long history, and it has always been one of the most important and challenging problems in analyzing and designing control systems. Particularly in the last two decades, the stabilizability problem in linear control systems of finite dimension has recieved a great deal of attention, and a number of useful results on the problem have been obtained [1]-[4].

On the other hand, stabilization of infinite dimensional systems has also been studied mainly in attempting to extend the finite dimensional results to the infinite dimensional case. However, it is,in general, considerably different from the finite dimensional case, and requires more sophisticated mathematical techniques [5]-[12]. In particular, stabilization by output feedback is much more complicated than that by state feedback, and therefore no concrete result on the

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output feedback case have been obtained.

This paper presents a preliminary work on the output feedback stabilizability problem for infinite dimensional systems. More precisely we will study the weak stabilizability by output feedback for contractive linear systems defined in Hilbert space. Our approach is based on the work by C. D. Benchimol [8] which gives necessary and sufficient conditions for the weak stabilizability by state feedback for contractive systems. The main result obtained in this paper is a theorem stating a necessary and sufficient condition for the weak stabilizability by output feedback.

In Section II, we formulate the weak stabilizability problem for linear systems defined in Hilbert space. Section III is devoted to the basic definitions and the known results which are relevant to the main body of this paper. Finally, Section IV presents the main results of this paper in which, under certain assumptions, a necessary and sufficient condition for the weak stabilizability by output feedback is proved.

II. The Stabilizability Problem.

The linear control system we will consider is described by the following abstract differential equation and output equation :

$$(2.1) \quad \mathcal{S} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \; x(0) = x_0 \text{ in } X \\ y(t) = Cx(t) \end{cases}$$

Here A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on a Hilbert space X (called the state space), B is a bounded linear operator from a Hilbert space U (the control space) into X, and finally C is a bounded linear operator from X into another Hilbert space Y (the output space). Throughout this paper, we denote for notational simplicity all the inner products and norms by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively, without specifying the Hilbert space in question.

(2.2) Definition. A state x in X is said to be weakly stable(w-stable) if $(T(t)x,z) \to 0$ as $t \to \infty$ for each z in X, and the set of all w-stable states, denoted by $X_{ws}(T)$, is called the w-stable subspace of $\mathcal{S}$. The system $\mathcal{S}$ is said to be w-stable if $X_{ws}(T) = X$. //

(2.3) Definition. A state x in X is said to be strongly stable(s-stable) if $\| T(t)x \| \to 0$ as $t \to \infty$, and the set of all s-stable states, denoted by $X_{ss}(T)$, is called the s-stable subspace of $\mathcal{S}$. The system $\mathcal{S}$ is said to be s-stable if $X_{ss}(T) = X$. //

It is easily seen that both $X_{ws}(T)$ and $X_{ss}(T)$ are closed subspaces of X. The orthogonal complement $X_{wus}(T)$ of $X_{ws}(T)$ will be called the weakly unstable subspace of $\mathcal{S}$, and similarly the orthogonal complement $X_{sus}(T)$ of $X_{ss}(T)$ the strongly unstable subspace of $\mathcal{S}$.

(2.4) Remark. We remark that the s-stability always implies the w-stability and that the w-stability is equivalent to the s-stability when any one of the following conditions is met [8]:

(i) X has a finite dimension.

(ii) A is a self-adjoint operator, i.e., T(t) are self-adjoint opeartors for all $t \geq 0$.

(iii) A has a compact resolvent, i.e., $(\lambda I - A)^{-1}$ is a compact operator for some λ in the resolvent set $\rho(A)$ of A.

(iv) T(t) are compact operators for all $t \geq 0$. //

We will now formulate the stabilizability problem. As usual, we denote by $\mathcal{B}(Z,W)$ the set of all bounded linear operators from Hilbert space Z into another Hilbert space W. For F in $\mathcal{B}(Y,U)$ we set u(t) =

$Fy(t) + v(t)$ ($= FCx(t) + v(t)$). Then the system $\mathcal{S}$ of (2.1) is reduced to the following system :

$$(2.5) \quad \mathcal{S}_F : \begin{cases} \dot{x}(t) = (A+BFC)x(t) + Bv(t), \ x(0) = x_0 \text{ in } X \\ y(t) = Cx(t) \end{cases}$$

This system $\mathcal{S}_F$ is usually refered to as the <u>output feedback system</u> of $\mathcal{S}$ with <u>output feedback gain</u> F. When the output $y(t)$ equals the state $x(t)$, i.e., $C = I$, the identity operator on X, $\mathcal{S}_F$ is called the <u>state feedback system</u>. Let us denote by $\{T_F(t); t \geq 0\}$ the C_0-semigroup generated by $A + BFC$. Then our stabilizability problem can be stated as follows.

(2.6) <u>The Stabilizability Problem</u>. The system $\mathcal{S}$ of (2.1) is said to be <u>weakly (strongly) stabilizable</u> if there exists an F in $\mathcal{B}(Y,U)$ so that the system $\mathcal{S}_F$ of (2.5) is weakly (strongly) stable. //

III. <u>Basic Definitions and Preliminaries</u>.

Let us denote by $L^2(S;Z)$ the set of all functions $f:S \to Z$ such that $\int_S \|f(t)\|^2 dt < \infty$ where S is an interval in the real line and Z is a Hilbert space. Now we begin with two basic definitions for the system $\mathcal{S}$ given in (2.1).

(3.1) <u>Definition</u>. A state x in X is said to be <u>controllable</u> if for any $\varepsilon > 0$ there exist a $t \geq 0$ and a $u(\cdot)$ in $L^2([0,t];U)$ such that

$$\| x - \int_0^t T(t-s)Bu(s)ds \| < \varepsilon .$$

The set of all controllable states, denoted by $X_c(A,B)$, is called the <u>controllable subspace</u> of $\mathcal{S}$, and $\mathcal{S}$ is said to be <u>controllable</u> if $X_c(A,B) = X$. //

(3.2) <u>Definition</u>. A state x in X is said to be <u>observable</u> if there exists a finite time $t \geq 0$ such that x is uniquely determined from $\{CT(s)x;\ 0 \leq s \leq t\}$. The set of all observable states, denoted by $X_0(A,C)$, is called the observable subspace of $\mathcal{S}$, and $\mathcal{S}$ is said to be observable if $X_o(A,C) = X$. //

We can easily prove the following proposition [11].

(3.3) Proposition. The controllable subspace $X_c($

subspace $X_o(A,C)$ of system $\mathcal{S}$ given in (2.1) are characterized by the following identities :

$$\text{(i)} \quad X_c(A,B) = \overline{\bigcup_{t \geq 0} T(t)BU}$$

$$\text{(ii)} \quad X_o(A,C) = \overline{\bigcup_{t \geq 0} T^*(t)C^*Y}$$

where " $*$ " and " $-$ " indicate the adjoint and the closure, respectively.//

(3.4) Remark. It follows from the above proposition that both subspaces $X_c(A,B)$ and $X_o(A,C)$ are closed. The orthogonal complement $X_{uc}(A,B)$ of $X_c(A,B)$ is called the uncontrollable subspace of $\mathcal{S}$, and is characterized by

$$\text{(i)} \quad X_{uc}(A,B) = \bigcap_{t \geq 0} \mathit{Ker}[B^*T^*(t)].$$

Similarly, the orthogonal complement $X_{uo}(A,C)$ of $X_o(A,C)$ is called the unobservable subspace of $\mathcal{S}$, and is characterized by

$$\text{(ii)} \quad X_{uo}(A,C) = \bigcap_{t \geq 0} \mathit{Ker}\,[CT(t)]. \quad //$$

Now let us mean $\mathcal{R}$ and $\mathcal{C}$ to be the real line and the complex plane, respectively, and $\mathcal{R}^q$ to be the q-dimensional Euclidean space. Furthermore, let us define the symmetric set $\mathcal{D}_n$ to be the set of all $\{\lambda_1,\cdots,\lambda_n\} \subset \mathcal{C}$ such that if λ_i is nonreal then some λ_j equals the complex conjugate $\overline{\lambda_i}$. As usual, the notation $\sigma(K)$ is used to mean the set of all eigenvalues of matrix K.

For a moment, assume that the system $\mathcal{S}$ of (2.1) is finite dimensional, i.e., assume $X = \mathcal{R}^n$, $Y = \mathcal{R}^m$ and $U = \mathcal{R}^r$. Then, $\mathcal{S}$ is said to be pole-assignable if for any $\{\lambda_1,\cdots,\lambda_n\}$ in $\mathcal{D}_n$ an $r \times n$ real matrix F can be found so that

$\sigma(A + BFC)$ is arbitrarily close to $\{\lambda_1, \cdots, \lambda_n\}$.

(3.5) Proposition. For the finite dimensional system S the following statements hold [2][4]:

(i) S is s-stable if and only if $\mathcal{Re}\,\lambda<0$ for all λ in $\sigma(A)$.

(ii) S is s-stabilizable by state feedback if and only if the strongly unstable subspace $X_{sus}(T)$ is contained in the controllable subspace $X_c(A,B)$.

(iii) S is pole-assignable by state feedback if and only if S is controllable and observable.

(iv) S is pole-assignable by output feedback if S is controllable and observable, and $n+1 \leq m+r$. //

(3.6) Remark. It is clear from the above proposition that pole-assignability implies s-stabilizability, but the converse does not hold.//

We will now return to the infinite dimensional case. The stabilizability problem of the system (2.1) has been studied in the Banach or Hilbert space framework in a number of recent papers, see e.g., [6]-[10]. The most relevant work to the present investigation is that of Benchimol [8], and his main result is cited below.

(3.7) Proposition. Let the C_0-semigroup $\{T(t)\,;\, t\geq 0\}$ of system (2.1) be contractive, i.e., $\|T(t)\| \leq 1$ for all $t \geq 0$. Then,

(i) S is w-stabilizable by state feedback if and only if the weakly unstable subspace $X_{wus}(T)$ is a subset of the controllable subspace $X_c(A,B)$, and

(ii) if S is w-stabilizable $F=-B^*$ is an w-stabilizing state feedback gain. //

From Remark (2.4) the following corollary is immediate.

(3.8) Corollary. Assume that $\{T(t)\,;\,t \geq 0\}$ be contractive. Then $\mathcal{S}$ is s-stabilizable by state feedback if and only if $X_{ss}(A,B) \supset X_{uc}(T)$, provided any one of the following conditions is met :

(i) A is self-adjoint.

(ii) A has a compact resolvent.

(iii) A generates a compact semigroup $\{T(t)\,;\,t \geq 0\}$. //

(3.9) Remark.

(i) It is worthwhile to note that a number of systems appearing in the practical applications satisfy one of the assumptions of (3.8). Therefore, studying w-stabilizability is of great importance from not only the mathematical interest but also its practical applicability.

(ii) Some sufficient conditions for s-stabilizability by state feedback have been obtained by Slemrod [6] and Levan and Rigby [10]

(iii) Triggani [7] has also discussed the state feedback s-stabilizability problem in the framework of Banach spaces.

(iv) The pole-assignability problem using state feedback has been investigated by Feintuch and Rosenfeld [9]. //

IV. The Weak Stabilizability by Output Feedback.

This section is concernd with the output feedback w-stabilizability problem. First we will prove a necessary condition for a linear control system to be w-stabilizable. Then under certain assumptions it will be shown that this condition is a necessary and sufficient condition for w-stabilizability of contractive systems. The main tool for this proof is Proposition (3.7).

Recalling that $\{T(t)\,;\,t \geq 0\}$ and $\{T_F(t)\,;\,t \geq 0\}$ are the semigruops associated with system $\mathcal{S}$ of (2.1) and its output feedback system $\mathcal{S}_F$, respectively, we will start with proving the following lemma.

(4.1) Lemma. For any output feedback gain F in $\mathcal{B}(Y,U)$, the following statements hold :

(i) The uncontrollable subspaces $X_{uc}(A,B)$ and $X_{uc}(A+BFC,B)$ are identical, and $T^*(t)x = T_F^*(t)x$ for all $t \geq 0$ and all x in $X_{uc}(A,B)$ $= X_{uc}(A+BFC,B)$.

(ii) The unobservable subspaces $X_{uo}(A,C)$ and $X_{uo}(A+BFC,C)$ are identical, and $T(t)x = T_F(t)x$ for all $t \geq 0$ and all x in $X_{uo}(A,C)$ $= X_{uo}(A+BFC,C)$.

(proof) We first note that the C_0-semigroups generated by the adjoint operators A^* and $(A+BFC)^*$ are equal to $\{T^*(t) \,;\, t \geq 0\}$ and $\{T_F^*(t) \,;\, t \geq 0\}$, respectively. Now consider the following equation :

$$\dot{\xi}(t) = A^*\xi(t) = (A+BFC)^*\xi(t) - C^*F^*B^*\xi(t)$$

from which we can easily deduce the identity

$$T^*(t)x = T_F^*(t)x - \int_0^t T_F^*(t-s)C^*B^*T^*(s)x\,ds \quad \text{for all x in X.} \qquad (1)$$

Suppose x belongs to $X_{uc}(A,B)$. Then by virtue of Remark (3.4,i) we have $B^*T^*(s)x = 0$ for all $s \geq 0$, and hence (1) implies

$$T^*(t)x = T_F^*(t)x \quad \text{for all } t \geq 0. \qquad (2)$$

So we get $B^*T_F^*(t)x = 0$ for all $t \geq 0$ proving $X_{uc}(A,B) \subset X_{uc}(A+BFC,B)$.

To show the reverse inclusion, consider

$$\dot{\xi}(t) = (A+BFC)^*\xi(t) = A^*\xi(t) + C^*F^*B^*\xi(t)$$

which gives the identity

$$T_F^*(t)x = T^*(t)x + \int_0^t T^*(t-s)C^*F^*B^*T_F^*(s)\,ds \quad \text{for all x in X.} \qquad (3)$$

Now we take x from $X_{uc}(A+BFC,B)$. Then employing the same argument as before yields $B^*T_F^*(s)x = 0$ for all $s \geq 0$, and (3) gives

$$T_F^*(t)x = T^*(t)x \quad \text{for all } t \geq 0. \qquad (4)$$

Hence $B^*T^*(t)x = 0$ for all $t \geq 0$, and the reverse inclusion $X_{uc}(A,B) \supset X_{uc}(A+BFC,B)$ obtains. This completes the proof of (i).

The statement (ii) can be shown in the same manner as in the proof of (i) except that this time we use the following equations :

$$\dot{\xi}(t) = A\xi(t) = (A+BFC)\xi(t) - BFC\xi(t)$$

$$\dot{\xi}(t) = (A+BFC)\xi(t) = A\xi(t) + BFC\xi(t)$$

which immediately give the identities

$$T(t)x = T_F(t)x - \int_0^t T_F(t-s)BFCT(s)xds \text{ for all } x \text{ in } X \tag{5}$$

$$T_F(t)x = T(t)x + \int_0^t T(t-s)BFCT_F(s)xds \text{ for all } x \text{ in } X. \tag{6}$$

Using Remark (3.4,ii), we can easily show the statement (ii). //

Now we use the above lemma to show the following theorem.

(4.2) Theorem. If the system $\mathcal{S}$ given in (2.1) is w-stabilizable by output feedback, then the w-stable subspace $X_{ws}(T)$ of $\mathcal{S}$ includes both the uncontrollable subspace $X_{uc}(A,B)$ and the unobservable subspace $X_{uo}(A,C)$.

(proof) Let F be a w-stabilizing gain of $\mathcal{S}$, and x be in $X_{uc}(A,B)$. Then by Lemma (4.1,i) we have $T^*(t)x = T_F^*(t)x$ for all $t \geq 0$, and thus for any z in X

$$(T^*(t)x,z) = (T_F^*(t)x,z) = (x,T_F(t)z) \to 0 \text{ as } t \to \infty$$

which implies that x belongs to $X_{ws}(T)$. So we obtain the inclusion $X_{ws}(T) \supset X_{uc}(A,B)$.

To show $X_{ws}(T) \supset X_{uo}(A,C)$, take an arbitrary element x from $X_{uo}(A,C)$. Then Lemma (4.1,ii) implies $T(t)x = T_F(t)x$ for all $t \geq 0$. Hence we obtain for any z in X

$$(T(t)x,z) = (T_F(t)x,z) \to 0 \text{ as } t \to \infty$$

which proves the desired inclusion $X_{ws}(T) \supset X_{uo}(A,C)$. //

Next we will prove the following lemma that plays a vital role to show our main theorem (4.4).

(4.3) <u>Lemma.</u> Let L be in $\mathcal{B}(X,U)$ and N be in $\mathcal{B}(X,Y)$ such that the range $\mathcal{R}(N)$ is closed in Y. Then $\mathcal{K}er\, L \supset \mathcal{K}er\, N$ if and only if there exists an F in $\mathcal{B}(Y,U)$ such that $FN = L$.

(proof) The sufficiency part is obvious. So, we prove only the necessity part.

We define $\pi_L : X \to X/\mathcal{K}er\, L$ by $\pi_L x = x + \mathcal{K}er\, L$, $\tilde{L}: X/\mathcal{K}er\, L \to \mathcal{R}(L)$ by $\tilde{L}(x + \mathcal{K}er\, L) = Lx$, and finally $\iota_L : \mathcal{R}(L) \to U$ by $\iota_L u = u$. Similarly define $\pi_N : X \to X/\mathcal{K}er\, N$, $\tilde{N} : X/\mathcal{K}er\, N \to \mathcal{R}(N)$, and $\iota_N : \mathcal{R}(N) \to Y$. Then we can easily see

$$Lx = \iota_L \tilde{L}\pi\, x = \tilde{L}\pi_L x \quad \text{for all } x \text{ in } X \tag{1}$$

$$Nx = \iota_N \tilde{N}\pi_N x = \tilde{N}\pi_N x \quad \text{for all } x \text{ in } X. \tag{2}$$

Since $\tilde{N}$ is bijective, we obtain

$$\tilde{N}^{-1}Nx = \tilde{N}^{-1}\tilde{N}\pi_L x = \pi_L x = x + \mathcal{K}er\, N. \tag{3}$$

Now define $\hat{L} : X/\mathcal{K}er\, N \to X/\mathcal{K}er\, L$ by $\hat{L}(x + \mathcal{K}er N) = \tilde{L}(x + \mathcal{K}er\, L)$. It should be noted that $\hat{L}$ is well defined since $\mathcal{K}er\, N \subset \mathcal{K}er\, L$ by assumption. Then by virtue of (1) and (3), we have

$$Lx = \iota_L \hat{L}\tilde{N}^{-1}Nx \qquad \text{for all } x \text{ in } X \tag{4}$$

It is not difficult to see that $\tilde{F} = \iota_L \hat{L}\tilde{N}^{-1}$ is a linear operator from $\mathcal{R}(N)$ into U. Since $\mathcal{R}(N)$ is a closed subspace of Y, it is meaningful to define a linear operator $F : Y \to U$ by

$$Fy = \begin{cases} \tilde{F}y = \iota_L \hat{L}\tilde{N}^{-1}y & \text{if } y \text{ in } \mathcal{R}(N) \\ 0 & \text{if } y \text{ in } \mathcal{R}(N)^{\perp} \end{cases}$$

where $\mathcal{R}(N)^{\perp}$ indicates the orthogonal complement of $\mathcal{R}(N)$. Then it follows from (4) that the $Lx = FNx$ for all x in X, i.e., $L = FN$. Moreover from the closedness of $\mathcal{R}(N)$ and the definition of F it is not difficult to check that F is bounded. This completes the proof. //

(4.4) Theorem. Assume for the system $\mathcal{S}$ of (2.1) that

(i) $\mathcal{S}$ is contractive, i.e., $\|T(t)\| \leq 1$ for all $t \geq 0$,

(ii) $\mathit{Ker}\, B^* \supset \mathit{Ker}\, C$, and

(iii) $\mathcal{R}(C)$ is closed.

Then, $\mathcal{S}$ is w-stabilizable by output feedback if and only if $X_{uc}(A,B) \subset X_{ws}(T)$ and $X_{uo}(A,C) \subset X_{ws}(T)$.

(proof) This theorem is an immediate consequence of Proposition(3.7), Theorem (4.2) and Lemma (4.3). //

From Remark (2.4) or Corollary (3.8) the following corollary immediately obtains.

(4.5) Corollary. Suppose that all the assumptions (i) – (iii) of (4.4) be satisfied. Then $\mathcal{S}$ is s-stabilizable by output feedback if and only if $X_{uc}(A,B) \subset X_{ws}(T)$ and $X_{uo}(A,C) \subset X_{ws}(T)$, provided any one of the following conditions is met :

(i) A is self-adjoint.

(ii) A has a compact resolvent.

(iii) A generates a compact semigroup $\{T(t) : t \geq 0\}$. //

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